On the choice of time in the continuum limit of polymeric effective theories

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Abstract

In polymeric quantum theories, a natural question pertains to the so called continuum limit, corresponding to the limit where the 'discreteness parameter' λ approaches zero. In particular one might ask whether the limit exists and, in that case, what the limiting theory is. Here we review recent results on the classical formulation of the problem for a soluble model in loop quantum cosmology. We show that it is only through the introduction of a particular λ-dependent internal time function that the limit $\lambda \to 0$ can be well defined. We then compare this result with the existing analysis in the quantum theory, where the dynamics was cast in terms of an internal (λ-independent) parameter for which the limit does not exist. We briefly comment on the steps needed to define the corresponding time parameter in the quantum theory for which the limit was shown to exist classically.

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I. INTRODUCTION

In the study of simple cosmological models from the perspective of loop quantum gravity, a non-standard, *polymeric* quantum theory [1] has proven to be essential [2]. One of the main features of this class of quantum theories is that they depend on a new dimensionful parameter λ. In the case of simple quantum mechanical systems like the harmonic oscillator [3, 4], this parameter has information about the discretization of space; the position operator becomes discrete in equidistant steps given by λ. In loop quantum cosmology (LQC) the parameter has information about the 'discreteness' of the loop geometry encoded in the fact that one considers finite holonomies to approximate the curvature of the connection. Since the standard Wheeler DeWitt (WDW) quantum theory can be recovered by setting $\lambda = 0$ in the quantum constraint, one might wonder what is the corresponding limit of dynamical physical states. Do we recover the WDW states in the limit?

In the case of a simple model in loop quantum cosmology, a spatially flat FRW model coupled to a massless scalar field, there has been a certain amount of results in this direction. In [5] the quantum model was shown to be exactly soluble and the issue of the limit $\lambda \to 0$ was addressed. In that case, the choice of internal time was rather natural since the physical quantum states could be interpreted as *evolving* with respect to an internal time given by the massless scalar field ϕ. It was shown there that the limit $\lambda \to 0$ was not uniform, *with respect to the internal time ϕ*, so the expectation value of the volume, as function of ϕ, does not approximate the corresponding value of the WDW theory for all times. One does not recover the WDW theory in this limit [5, 6]. The question remained open as to whether the limit exists for a different choice of internal time.

A simpler and conceptually less sophisticated question pertains to the corresponding limit in the classical domain. There, a one parameter family of classical constrained systems, labeled by λ, exists. They contain the dynamical information of semiclassical states of the quantum theory, and have sometimes been referred to as the *effective theory*. In other instances one could simply consider a family of such classical theories, without reference to a particular quantum theory. It is then natural to ask whether the limiting process $\lambda \to 0$ is well defined for these theories. Borrowing the nomenclature from the quantum theory, we refer to this limit as the *continuum limit* of the effective theories (see [7] for further discussion on this issue).

Recently, this problem involving classical theories was studied in detail [7]. It was argued that, in order to have a well defined limit, one needs to introduce an *internal time function* as a physical Dirac observable with respect to which the dynamics can be described. Furthermore, the corresponding time function has to depend explicitly on the parameter λ for the limit to be well defined, and the dependence is severely limited. In particular, it was shown that, for the LQC example, the internal time given by the scalar field is not a good parameter for this purposes. That is, the limit is not well defined in that case, in complete analogy with the quantum result of [5].

The purpose of this contribution is to make the relation between these two results transparent. In the first part we review the classical theory and explicitly construct the preferred time function that, interestingly, turns out to be the parameter that, in the spacetime interpretation, measures proper time of co-moving observers. We show that with respect to proper time, the limit is well defined. In the second part we briefly review the quantum theory and explicitly show the λ-dependence of the expectation value of the volume operator as a function of the scalar field ϕ. We compare the expectation values on different scales

and give a map between the corresponding states for which the expectation values are close to each other for large positive values of the scalar field, but then they grow apart as ϕ gets smaller. A different map is needed to 'synchronize' the expectation values for large negative ϕ, but as ϕ goes away from this asymptotic region the expectation values get further apart. We can not simultaneously synchronize the both branches, because the bounce occurs for different values of ϕ at each scale. We also show that, taking the limit $\lambda \to 0$, one can indeed recover either branch of the WDW theory (contracting or expanding) from the LQC states and approximate the dynamics (as described by expectation values) far from the bounce. But, as noted earlier [5, 6], one can not recover the whole dynamical trajectory since the LQC dynamics has always a bounce, while the WDW theory follows the classical trajectory into the singularity.[1] We end with some comments on the open problem of how to define proper time in the quantum theory with respect to which the convergence question can be posed.

II. CLASSICAL THEORY

We are interested in completely constrained system with one constraint $\mathcal{C}$, that generates the time evolution, which is actually a gauge transformation. We denote by $\mathcal{M}$ the $2n$ dimensional phase space of this system, with canonical coordinates $\{q^a, p_a\}$, $a = 1, \ldots n$. For this system we need to solve the constraint $\mathcal{C}(q^a, p_a) \approx 0$ and to find gauge invariant phase space functions, i.e. Dirac observables. We shall follow the ideas of Rovelli and Dittrich, who associate to every pair of phase space functions a one-parameter family of Dirac observables [10, 11]. This construction starts with the definition of the flow α_C^t generated by the constraint, where t is an evolution parameter associated to $\mathcal{C}$. For an arbitrary smooth phase space function $F(q^a, p_a)$ it can be calculated as

$$\alpha_C^t(F) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \{\mathcal{C}, F\}_n \,, \tag{1}$$

where $\{\mathcal{C}, F\}_0 = F$ and $\{\mathcal{C}, F\}_{n+1} = \{\mathcal{C}, \{\mathcal{C}, F\}_n\}$.

A partial observable is an arbitrary phase space function, to which in principle one can associate a number after some measuring procedure. It does not have to be a Dirac observable. We first choose as a partial observable the phase space function $R(q_a, p_a)$ that can be used to measure the "time" along the gauge orbits, generated by the constraint. In order to be a good "clock" variable the function $R(t) \equiv \alpha_C^t(R)$ has to be invertible. We shall consider another phase space function $F(q^a, p_a)$ and calculate its value when $R(t) = t_0$

$$F|_{t_0} \equiv \alpha_C^t(F)|_{R(t)=t_0} \,. \tag{2}$$

Since $R(t)$ is an invertible function, $F|_{t_0}$ form a one parameter family of (complete) Dirac observables, i.e. $\{F|_{t_0}, \mathcal{C}\} = 0$ (For details, see [11]). It is obvious that we can choose different "clock" variables, and the resulting family of Dirac observables, in general, are not equivalent. We are going to exploit this idea and analyze two different choices for $R(q_a, p_a)$ and the corresponding Dirac observables and we shall argue that one of them

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[1] This has to be contrasted with some simple unconstrained quantum mechanical systems where the limit has been shown to exist [3, 4, 8, 9].

is more convenient in order to study the convergence of the suitable constructed effective theories.

As our first example of a totally constrained classical system we shall consider a $k = 0$ FRW cosmological model coupled to a massless scalar field, and we shall construct one family of Dirac observables. In this case the metric takes the form

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \delta_{ij} dx^i dx^j \,, \tag{3}$$

where $N(t)$ is (arbitrary) lapse function and $a(t)$ is the scale factor. The classical phase space for the gravitational sector can be described by two variables β and V, where, on shell, β is proportional to the Hubble parameter, $\beta = \gamma \frac{\dot{a}}{a}$ (γ is the Barbero-Immirzi parameter) and V is the physical volume of the cell $\mathcal{V}$. Note that $\beta \in (-\infty, \infty)$ and $V \geq 0$. The pair of conjugate variables satisfy $\{\beta, V\} = 4\pi G\gamma$. The matter sector is given by a scalar field ϕ and it momentum p_ϕ, such that $\{\phi, p_\phi\} = 1$.

This classical system is subject to the Hamiltonian constraint [2]:

$$\mathcal{C} = N\left(-\frac{3}{8\pi G\gamma^2} V\beta^2 + \frac{p_\phi{}^2}{2V}\right) \,. \tag{4}$$

If we choose $N = V$ the constraint takes simpler, classically equivalent form

$$\tilde{\mathcal{C}} = -\frac{3}{8\pi G\gamma^2} V^2 \beta^2 + \frac{p_\phi^2}{2} \,. \tag{5}$$

The momentum p_ϕ is a Dirac observable and therefore, a constant of motion (we choose $p_\phi > 0$ without loss of generality). In order to construct one family of complete observables we should choose a suitable "clock" phase space function. From the form of the constraint $\tilde{\mathcal{C}}$ we can see that ϕ is a good choice. Namely,

$$\alpha^t_{\tilde{C}}(\phi) = \phi - p_\phi t \,, \tag{6}$$

is an invertible function of t. For the other partial observable we choose V and calculate its flow, it turns out that it can be written in a closed form

$$\alpha^t_{\tilde{C}}(V) = V \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{3}{\gamma} V\beta t\right)^n = V e^{-\frac{3}{\gamma} V\beta t} \,. \tag{7}$$

Since from $\alpha^t_{\tilde{C}}(\phi) = \phi_0$ it follows that $t = \frac{\phi - \phi_0}{p_\phi}$, and from (5) we obtain that $\frac{3}{\gamma} V|\beta| \approx \kappa p_\phi$, where $\kappa = \sqrt{12\pi G}$, we can express the corresponding one parameter family of complete observables as

$$V|_{\phi_0} = V e^{-\kappa(\operatorname{sgn}\beta)(\phi - \phi_0)} \,. \tag{8}$$

As our next example we shall consider a class of so called 'effective theories' characterized by a parameter λ with dimension of length, which can be thought of as a 'polymeric extension' of the system (5). This LQC model that has received some of attention given that it can be solved both classical and quantum mechanically, for arbitrary values of λ. For details of the model see [5]. The origin of λ in the context of quantum theory is the discretization of $\hat{V}$, and we take λ as a free parameter and focus on the λ dependence of the corresponding

Dirac observables. The effective theory on the scale λ is defined by the constraint [2, 5] (we again take $N = V$)

$$\tilde{\mathcal{C}}_\lambda = -\frac{3}{8\pi G\gamma^2}\frac{1}{\lambda^2}V^2 \sin^2(\lambda\beta) + \frac{p_\phi^2}{2} , \tag{9}$$

where β can be seen as being compactified, taking values in $\beta \in (-\frac{\pi}{2\lambda}, \frac{\pi}{2\lambda}]$. This constraint implies that there is a minimum value of V at every scale, $V_{b,\lambda} = \frac{\kappa}{3}p_\phi\gamma\lambda$ attained when $\beta = \frac{\pi}{2\lambda}$. It turns out that all gauge trajectories have a bounce at a point $(V, \beta, \phi, p_\phi)_{\lambda,b} = (\frac{\kappa}{3}p_\phi\gamma\lambda, \frac{\pi}{2\lambda}, \phi_{\lambda,b}, p_\phi)$, which depends on λ and p_ϕ. We note that $\phi_{\lambda,b}$ is a finite number, that behaves as $\ln\lambda$ as $\lambda \to 0$ [7]. This should be contrasted with the behavior of the gauge trajectories obtained from (5), where there is no bounce and $V \to 0$ as $\beta \to \infty$.

We note that $\lim_{\lambda\to 0}\tilde{\mathcal{C}}_\lambda = \tilde{\mathcal{C}}$. We are going to construct a set of complete observables at every scale and analyze their behavior as $\lambda \to 0$, to be able to gain more insight into the relation between the $k = 0$ FRW cosmological model and the effective theories described above. As before, we take ϕ and V as two partial observables, and consider their flows. The flow of ϕ is the one that we obtained previously, while the flow of V takes the form

$$\alpha^t_{\tilde{\mathcal{C}}_\lambda}(V) = V\left[\sinh\left(-\frac{3}{\gamma\lambda}Vt\sin\lambda\beta\right)\cos\lambda\beta + \cosh\left(\frac{3}{\gamma\lambda}Vt\sin\lambda\beta\right)\right] , \tag{10}$$

resulting in the following family of Dirac observables

$$V_\lambda|_{\phi_0} = V_\lambda\left[\sinh\left(-\kappa(\mathrm{sgn}\beta)(\phi - \phi_0)\right)\cos\lambda\beta + \cosh\left(\kappa(\mathrm{sgn}\beta)(\phi - \phi_0)\right)\right] , \tag{11}$$

where the notation V_λ indicates that we are at the scale λ.

It can be shown easily [7] that if we consider a gauge trajectory $V = V(\phi)$ in FRW model and compare it to $V = V_\lambda(\phi)$ it is always possible to choose some ϕ_0 when the two trajectories are close to each other and than see what happens during the evolution of the two systems. The two trajectories start close to each other but start to diverge as one approaches the bounce of the effective theory. As we make λ smaller we can enlarge the interval of convergence, but eventually the two start to diverge. We can also see this behavior at the level of Dirac observables. We see that $V_\lambda|_{\phi_0}$ at bounce ($\beta = \frac{\pi}{2\lambda}$) behaves as $V\cosh\left(\kappa(\mathrm{sgn}\beta)(\phi - \phi_0)\right)$ which should be compared to (8). Of course it is just another point of view on the convergence behavior that we already described at the level of gauge orbits.

Since the choice of a "clock" variable is not unique we can explore this freedom to search for the set of complete observables that has better convergence properties than the one already constructed. One possible "natural" candidate that represents a good parametrization of the gauge orbits is the cosmological proper time τ. In this case we fix the lapse function as $N = 1$ and one of the corresponding equations of motion $\frac{d\tau}{d\phi} = \frac{V}{p_\phi}$ that, combined with the rest of the equations can be solved leading to

$$\tau = (\mathrm{sgn}\beta)e^{\kappa(\mathrm{sgn}\beta)\phi} , \tag{12}$$

where the integration constant is fixed by the condition that $V(\tau) \to 0$ as $\tau \to 0$. In this way we can construct τ as a function on the phase space and choose it as the partial observable that represents the "clock" function. We note that $\tau \neq 0$ on the constraint surface. The flow of τ is

$$\alpha^t_{\tilde{\mathcal{C}}}(\tau) = \tau e^{-\kappa(\mathrm{sgn}\beta)p_\phi t} , \tag{13}$$

and the one parameter family of complete observables is given by

$$V|_{\tau=\tau_0} = V \frac{\tau_0}{\tau} \, . \tag{14}$$

So, there are two sets of values that this observables can take, depending on the sign of τ. One of them, $\tau > 0$, corresponds to the expanding universe, while the other one, for $\tau < 0$ corresponds to the universe in contraction.

We can use the same idea to construct new "clock", the λ-dependent partial observable in the effective theory

$$\tau_\lambda = \frac{\gamma\lambda}{3} \sinh\left(\kappa(\mathrm{sgn}\beta)\phi + \ln\frac{3}{\gamma\lambda} \right), \tag{15}$$

where the constant is fixed by the condition $\tau_\lambda \to \tau$ as $\lambda \to 0$. Note that at every scale the bounce occurs for the same value $\tau_\lambda = 0$. The flow of this function is

$$\alpha^t_{\hat{C}_\lambda}(\tau_\lambda) = \frac{\gamma\lambda}{3} \sinh\left(\kappa(\mathrm{sgn}\beta)(\phi - p_\phi t) + \ln\frac{3}{\gamma\lambda} \right). \tag{16}$$

When we calculate the family of corresponding complete observables we find that

$$V_\lambda|_{\tau_\lambda=\tau_0} = \frac{1}{2}V\left[(z - \frac{1}{z}) \cos\lambda\beta + (z + \frac{1}{z}) \right], \tag{17}$$

with $z = (3\tau_0 + \sqrt{\gamma^2\lambda^2 + 9\tau_0^2})(3\tau_\lambda + \sqrt{\gamma^2\lambda^2 + 9\tau_\lambda^2})^{-1}$. We can now compare the observables (17) at the bounce, which now occurs for the same value of $\tau_\lambda = 0$. In the limit $\lambda \to 0$ they converge uniformly. At every scale λ the gauge trajectory $V = V_\lambda(\tau_\lambda)$ is a hyperbola, whose asymptotes are two gauge trajectories of a FRW model; two half-lines that correspond to the expanding ($p_\phi > 0$) and contracting ($p_\phi < 0$) branches [7]. The limit of the gauge trajectories when $\lambda \to 0$ exists, but is a non-differentiable curve. We can conclude that the new observables (14) and (17) are more suited for the description of the convergence of the effective theories. The limiting theory is a "singular bouncing" universe and *not* the classical GR dynamics, because $V_\lambda(0) \to 0$ as $\lambda \to 0$, and it does not belong to the phace space of the $k = 0$ FRW cosmological model.

III. SOME RESULTS FROM THE QUANTUM THEORY

The purpose of this section is to explore the relation between the Wheeler DeWitt and LQC when analyzing the dynamics of the volume as function of the internal time ϕ.

Let us start with a very brief introduction of the quantum systems and their solutions, the details can be found in [5]. The WDW theory of the quantum $k=0$ FRW model coupled to a massless scalar field is obtained after the standard Schroedinger quantization of (5). In the (β, ϕ) representation the Hamiltonian constraint takes the form

$$\partial_\phi^2 \underline{\Psi}(\beta, \phi) = 12\pi G(\beta\partial_\beta)^2 \underline{\Psi}(\beta, \phi) \, . \tag{18}$$

Physical states must satisfy this equation, and from the form of the equation it follows that the scalar field can be used as a "clock" variable, just like in the classical theory. This constraint can be rewritten as the Klein-Gordon equation if we replace β by $y \equiv \frac{1}{\kappa} \ln \frac{\beta}{\beta_0}$, with

β_0 arbitrary. Different choices of β_0 yield unitarily equivalent quantum theories. We note that the change $\beta_0 \rightarrow \beta_0' = \alpha\beta_0$ (where α is an arbitrary real number) corresponds to the change $y \rightarrow y' = y - \frac{1}{\kappa} \ln \alpha$. The physical states take the form $\underline{\Psi}(y, \phi) = \underline{\Psi}_L(y_+) + \underline{\Psi}_R(y_-)$, where $y_\pm = \phi \pm y$. We will show later that left (right) moving states, $\underline{\Psi}_L(y_+)$ ($\underline{\Psi}_R(y_-)$) correspond to expanding (contracting) universes.

On the other hand, the solvable loop quantum cosmology (sLQC) model is obtained in the process of polymeric quantization of (5), where the discreteness parameter λ is introduced in order to calculate curvature in terms of holonomies [5] (Note that in this article the quantum theory was obtained by the standard loop quantization procedure without 'polymerization'). One could also interpret sLQC as being obtained from the standard Schroedinger quantization of the effective constraint (9) (where the variable β is taken to be periodic). In the (β, ϕ) representation the Hamiltonian constraint in sLQC takes the following form

$$\partial_\phi^2 \chi(\beta, \phi) = 12\pi G \left(\frac{\sin \lambda\beta}{\lambda} \partial_\beta \right)^2 \chi(\beta, \phi) . \tag{19}$$

We note that the physical states should be symmetric under the change of orientation of the physical co-triads. As was shown in [5], for the WDW states this condition implies that we can restrict the analysis to the positive β. For that reason the domain of β in sLQC is chosen to be $(0, \frac{\pi}{\lambda})$. [2]

The constraint (19) can also be transformed to a Klein-Gordon equation if we introduce a new λ-dependent variable $x = \frac{1}{\kappa} \ln \left(\tan \frac{\lambda\beta}{2} \right)$. The physical states in sLQC are $\chi(x, \phi) = \chi_L(x_+) + \chi_R(x_-)$, where $x_\pm = \phi \pm x$. The condition that the states should be symmetric under the change of the orientation of the physical co-triads leads to the restriction $\chi(-x, \phi) = -\chi(x, \phi)$. As a result, the sLQC states are of the form $\chi(x, \phi) = \frac{1}{\sqrt{2}} (F(x_+) - F(x_-))$, where F is some function (for details see [5]).

In order to get the physical content of these theories we need to define the complete set of physical observables. One possible choice in this case is $\hat{p}_\phi$ and $\hat{V}|_\phi$, in analogy with classical theory. In the WDW theory these observables are defined in the following way

$$\hat{p}_\phi \underline{\Psi}(y, \phi) = -i\partial_\phi \underline{\Psi}(y, \phi) = \sqrt{\underline{\Theta}} \, \underline{\Psi}(y, \phi) , \tag{20}$$

$$\hat{V}|_{\phi_0} \underline{\Psi}(y, \phi) = e^{i\sqrt{\underline{\Theta}}(\phi - \phi_0)} \hat{V} \, \underline{\Psi}(y, \phi_0) , \tag{21}$$

where $\underline{\Theta} \equiv -\partial_y^2$. The same set of Dirac observables can be constructed in sLQC, their form can be obtained from the above expressions after the substitution $\underline{\Psi}(y, \phi) \rightarrow \chi(x, \phi)$ and $\underline{\Theta} \rightarrow \Theta \equiv -\partial_x^2$.

Now, we want to compare the expectation values of $\hat{V}|_\phi$ in the appropriately chosen states in WDW theory and sLQC. In WDW for left moving states one obtains [5]

$$(\underline{\Psi}_L, \hat{V}|_\phi \underline{\Psi}_L)_{\text{phys}} = \tilde{V}_{0L} \, e^{\kappa\phi} , \tag{22}$$

where $\tilde{V}_{0L}$ is a constant depending on the initial conditions. The expectation value tends to infinity as $\phi \rightarrow \infty$, for any left moving state, so that this sector corresponds to an expanding universe. Similarly, for the right moving solutions

$$(\underline{\Psi}_R, \hat{V}|_\phi \underline{\Psi}_R)_{\text{phys}} = \tilde{V}_{0R} \, e^{-\kappa\phi} , \tag{23}$$

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[2] In the classical $k = 0$ FRW model we have $\beta \in (-\infty, \infty)$ and for that reason we have chosen a symmetric domain for β in the classical effective theories, namely, $\beta \in (-\frac{\pi}{2\lambda}, \frac{\pi}{2\lambda}]$.

and the right moving sector corresponds to a contracting universe. The explicit form of the constants is

$$\tilde{V}_{0(L,R)} \equiv \frac{s}{\beta_0} \int_{-\infty}^{\infty} \mathrm{d}y \left| \frac{\mathrm{d}\Psi_{(L,R)}(y)}{\mathrm{d}y} \right|^2 e^{\mp\kappa y} , \tag{24}$$

where $s = \frac{8\pi\gamma l_{\mathrm{Pl}}^2}{\kappa}$.

In sLQC the expectation value of $\hat{V}|_{\phi}$ at the state $\chi(x,\phi)$ is of the form

$$(\chi, \hat{V}|_{\phi}\chi)_{\mathrm{phy}} = \tilde{V}_+^{(\lambda)} e^{\kappa\phi} + \tilde{V}_-^{(\lambda)} e^{-\kappa\phi} , \tag{25}$$

where $\tilde{V}_\pm^{(\lambda)}$ depend on the initial data and also on λ

$$\tilde{V}_\pm^{(\lambda)} \equiv \frac{s\lambda}{2} \int_{-\infty}^{\infty} \mathrm{d}x \left| \frac{\mathrm{d}F(x)}{\mathrm{d}x} \right|^2 e^{\mp\kappa x} . \tag{26}$$

The volume expectation value has a non-zero minimum value

$$V_b^{(\lambda)} = 2\frac{\sqrt{\tilde{V}_+^{(\lambda)}\tilde{V}_-^{(\lambda)}}}{\|\chi_\lambda\|^2} , \quad \text{for} \quad \phi_b^{(\lambda)} = \frac{1}{2\kappa} \ln \frac{\tilde{V}_-^{(\lambda)}}{\tilde{V}_+^{(\lambda)}} , \tag{27}$$

and it is symmetric about the bounce point $\langle \hat{V}|_{\phi} \rangle_{\phi_b+\phi} = V_b \cosh \kappa\phi$.

Let us first compare the predictions on two scales λ and λ'. Let us suppose, without loss of generality, that $\lambda' < \lambda$. Since $x' \approx x - \frac{1}{\kappa} \ln \frac{\lambda}{\lambda'}$ for small β (that classically corresponds to the region where the universe is large), we can relate the states in two theories at different scales such that, for $\phi = 0$, satisfy the condition

$$F_{\lambda'}(x') = F_\lambda(x' + \nu) , \tag{28}$$

where $\nu = \frac{1}{\kappa} \ln \frac{\lambda}{\lambda'} > 0$. These states have the same norm and the same expectation value for any power of p_ϕ. The relation between their volume expectation values can be obtained from

$$\tilde{V}_\pm^{(\lambda')} = \frac{s\lambda'}{2} \int_{-\infty}^{\infty} \mathrm{d}x' \left| \frac{\mathrm{d}F_{\lambda'}(x')}{\mathrm{d}x'} \right|^2 e^{\mp\kappa x'} = \frac{s\lambda'}{2} \int_{-\infty}^{\infty} \mathrm{d}x' \left| \frac{\mathrm{d}F_\lambda(x' + \nu)}{\mathrm{d}x'} \right|^2 e^{\mp\kappa x'} = \frac{\lambda'}{\lambda} e^{\pm\kappa\nu} \tilde{V}_\pm^{(\lambda)}$$

so that

$$\tilde{V}_+^{(\lambda')} = \tilde{V}_+^{(\lambda)} \quad \text{and} \quad \tilde{V}_-^{(\lambda')} = \left(\frac{\lambda'}{\lambda} \right)^2 \tilde{V}_-^{(\lambda)} . \tag{29}$$

Thus, with this prescription we are 'synchronizing' the states in such a way that the expanding branch coincides for all values of the parameter λ. Clearly one should expect that these states approximate a WDW state for large positive values of ϕ. Note also that, for $\phi = 0$,

$$\langle \hat{V}|_0 \rangle^{(\lambda)} - \langle \hat{V}|_0 \rangle^{(\lambda')} = \left[1 - \left(\frac{\lambda'}{\lambda} \right)^2 \right] \tilde{V}_-^{(\lambda)} ,$$

and the volume expectation values on two scales are close to each other for large ϕ but they grow apart as $\phi \to \phi_b$ (and even further for $\phi < \phi_b$).

The minimum values of volume are related as

$$V_b^{(\lambda')} = \frac{\lambda'}{\lambda} V_b^{(\lambda)}, \quad \text{where} \quad \phi_b^{(\lambda')} = \phi_b^{(\lambda)} - \nu. \tag{30}$$

We see that $\phi_b^{(\lambda')} \to -\infty$ as $\lambda' \to 0$. As we have synchronized the states in such a way that the dynamics coincides for large positive values of the scalar field, what we are doing is to 'push' the bounce to more negative values of the internal time as we decrease the value of λ'. Let us now see how we can 'synchronize' states during the contracting phase, in such a way that now the bounce gets 'pushed' to large values of the internal time ϕ.

For that, let us choose the states such that, for $\phi = 0$,

$$F_{\lambda'}(x') = F_\lambda(x' - \nu), \tag{31}$$

we obtain, following the same steps as above, that

$$\tilde{V}_+^{(\lambda')} = \left(\frac{\lambda'}{\lambda}\right)^2 \tilde{V}_+^{(\lambda)} \quad \text{and} \quad \tilde{V}_-^{(\lambda')} = \tilde{V}_-^{(\lambda)}, \tag{32}$$

and we see that now the expectation values are close for large negative ϕ and they grow apart as ϕ approaches the corresponding bounce point. In this case

$$\phi_b^{(\lambda')} = \phi_b^{(\lambda)} + \nu, \tag{33}$$

and $\phi_b^{(\lambda')} \to \infty$ as $\lambda' \to 0$, which is what we wanted to get. Let us now compare sLQC states with those of the Wheeler DeWitt theory.

In order to be able to compare predictions from WDW theory and sLQC we should choose the states in both theories that are 'close' to each other for small β, in the sense that the expectation values of $\hat{V}|_\phi$ in these states should be close to each other. As we shall see, one can define such prescription for both branches of the WDW theory. First we should initially (for $\phi = 0$) choose a state $\underline{\Psi}_L(y)$ such that $\tilde{V}_-$ is small, when calculated on this state. In order to construct the corresponding state in sLQC we first note that

$$x \approx \frac{1}{\kappa} \ln \frac{\lambda\beta}{2} = y - \frac{1}{\kappa} \ln \frac{2}{\lambda\beta_0}, \quad \text{when} \quad \beta \to 0. \tag{34}$$

As a result, if we consider states such that

$$F_\lambda(x) \equiv \underline{\Psi}_L(x + \mu), \tag{35}$$

where $\mu = \frac{1}{\kappa} \ln \frac{2}{\lambda\beta_0}$, then the volume expectation values in these states are close. That is, for $\phi = 0$, they can be arbitrarily close for an appropriate choice of λ. We also note that these states have the same norm and that the corresponding expectation values of $\hat{p}_\phi$ are the same [5]. Using this result we can calculate the explicit λ dependence of $\tilde{V}_\pm^{(\lambda)}$,

$$\tilde{V}_\pm^{(\lambda)} = \frac{s\lambda}{2} e^{\pm\kappa\mu} \int_{-\infty}^{\infty} dx \left|\frac{d\underline{\Psi}_L(x)}{dx}\right|^2 e^{\mp\kappa x} \equiv \lambda e^{\pm\kappa\mu} K_\pm, \tag{36}$$

(notice that $K_\pm$ do not depend on λ), so that

$$\tilde{V}_+^{(\lambda)} = \tilde{V}_{0L}, \quad \tilde{V}_-^{(\lambda)} = \lambda^2 \frac{\beta_0^2}{4} \tilde{V}_{0R} \tag{37}$$

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we can now see that $\tilde{V}_-^{(\lambda)} \to 0$ as $\lambda \to 0$, so that for any ϕ fixed, the dominant part of (25) is the same as (22) in this limit.

Similarly, if we initially choose a state $\underline{\Psi}_R(y)$ such that $\tilde{V}_+$ is small, than the condition

$$F_\lambda(x) \equiv \underline{\Psi}_R(x - \mu)\,, \tag{38}$$

leads to

$$\tilde{V}_+^{(\lambda)} = \lambda^2 \frac{\beta_0^2}{4} \tilde{V}_{0L}\,, \quad \tilde{V}_-^{(\lambda)} = \tilde{V}_{0R}\,, \tag{39}$$

so, in this case $\tilde{V}_+^{(\lambda)} \to 0$ as $\lambda \to 0$. We can see that there is a convergence of the coefficients of (26) as $\lambda \to 0$ to their WDW counterparts (24), but the convergence of the expectation values (25) in non-uniform, due to a divergent behavior in the limit $\phi \to \pm\infty$.

Motivated by the results of the classical analysis presented in the previous section, we could try to construct another set of complete observables in quantum theory that would have better convergence properties in the limit $\lambda \to 0$. This could be $\hat{V}|_\tau$. In order to construct these observables we should first introduce the proper time operator on the corresponding Hilbert space, and then construct physical operators corresponding to, say, volume at a given proper time. This is a non trivial task that we leave for future work.

IV. DISCUSSION

The problem of taking the continuum limit of constrained polymeric quantum theories and loop quantum cosmology in particular is an open and interesting question. We have available some partial answers, that we have revisited in this contribution. On the one hand, a classical analysis of the related "effective theory" clearly shows that the issue of convergence depends very strongly on the choice of time. Only with respect to a particular λ-dependent internal time can the dynamics have a chance to converge. For the system under consideration we have shown that the cosmic proper time is a valid choice with respect to which the limit can be studied. As we saw, even when the limit exists and is well defined, the limiting dynamics is non-differentiable. That is, the resulting limit does not correspond to a cosmology satisfying reasonable equations of motion but rather suffers from a "singular bounce".

On the quantum front, the result that is available pertains to the evolution of the expectation value of volume with respect to the scalar field ϕ that can be used as an internal time. As we showed in detail, from the loop quantum theory one can indeed recover both the contracting and expanding branches of the Wheeler DeWitt theory, but strictly speaking, there is no convergence between the theories since the limit is not uniform. One should note that this result in the quantum domain is fully consistent with the classical analysis in which it was shown that using the scalar field as the internal time does not lead to convergence of the limit $\lambda \to 0$ for the effective theories.

The step that is clearly missing in order to have a converging quantum theory is to translate the results of the classical effective theory to the quantum theory. That is, one would like to have a consistent procedure to define the quantum dynamics in terms of proper time. The difficulty with this is that proper time is itself a rather non-trivial function on phase space, for which one needs to construct a well defined operator (See, for instance, [12] for a proposal). Furthermore, one needs to define a relational dynamics for the volume in terms of the proper time operator. One possibility would be to define the relational dynamics

in terms of expectation values. For instance, for the volume operator that one uses to explore singularity resolution in LQC, one could consider the evolution of the expectation value of the volume operator at a given value of the expectation value of the *proper time* operator. One would expect that quantity to be well defined in the limit, for the model we have here studied.

We shall leave a detailed analysis of the problem of defining the continuum limit for quantum constrained systems and the analysis of these simple systems for a future publication.

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